Solutions

Chapter 3
Differentiation Worksheet

Basic Rules for Differentiation

(1) $\frac{d}{dx}(cf(x)) = cf'(x)$

(2) $\frac{d}{dx}(f \pm g(x)) = f'(x) \pm g'(x)$

(3) $\frac{d}{dx}(x^n) = nx^{n-1}$

(4) $\frac{d}{dx}(e^{kx}) = ke^{kx}$

(5) $\frac{d}{dx}(a^x) = \ln(a)a^x$

(6) $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$

Exercises: Find the derivative of the given function unless otherwise specified. Assume that a, b, c, k are constants.

1. $y = 3t^4$
2. $h(\theta) = \frac{1}{\sqrt[3]{\theta}}$
3. $Q = aP^2 + bP^3$
4. $w = 3ab^2q$
5. $y = ce^t + k\ln t$
6. $P = 50e^{-0.06t}$
7. $w = 2^x + \frac{2}{x^3}$
8. $R(q) = q^2 + 2\ln(q)$
9. $P(t) = be^t$

(1) $\frac{dy}{dt} = 12t^3$ (2) $h'(\theta) = -\frac{1}{3}\theta^{-4/3} = \frac{-1}{3\sqrt[3]{\theta^4}}$ (3) $Q' = 2aP + 3bP^2$

(4) $\frac{dw}{dq} = 3ab^2$ (5) $\frac{dy}{dt} = ce^t + \frac{k}{t}$ (6) $\frac{dP}{dt} = -3e^{-0.06t}$

(7) $\frac{dw}{dx} = \ln 2 \cdot 2^x - \frac{6}{x^4}$ (8) $R'(q) = 2q + \frac{2}{q}$ (9) $P'(t) = be^t$

10. Find the equation to the line tangent to the graph of $f(x) = 2x^3 - 5x^2 + 3x - 5$ at $x = 1$.

$f'(x) = 6x^2 - 10x + 3$

$f'(1) = -1$ $f(1) = -5$

Thus $y = -x - 4$

11. Find the equation to the line tangent to the graph of $y = \ln x$ at $x = 1$ and use it to approximate $\ln(1.1)$.

$\frac{dy}{dx} = \frac{1}{x}$ $\left.\frac{dy}{dx}\right|_{x=1} = 1$ and $\ln 1 = 0$

Thus $\ln(1.1) \approx 0.1$

Chain Rule: For functions $y = f(z)$ and $z = g(t)$ the derivative of $y = f(g(t))$ is given by

$$\frac{d}{dt}(f(g(t))) = f'(g(t)) \cdot g'(t)$$

or

$$\frac{dy}{dt} = \frac{dy}{dz} \cdot \frac{dz}{dt}.$$

Combining this rule with the previous rules, if z is a differentiable function of t then

$$\frac{d}{dt}(z^n) = nz^{n-1}\frac{dz}{dt}, \qquad \frac{d}{dt}(e^z) = e^z\frac{dz}{dt}, \qquad \frac{d}{dt}(\ln z) = \frac{1}{z}\frac{dz}{dt}.$$

Exercises: Find the derivative of the given function unless otherwise specified.

12. e^{-x^2} 13. $(x^2+4)^3$ 14. $5\ln(2t^2+3)$

15. $f(\theta) = (e^\theta + e^{-\theta})^{-1}$ 16. $f(x) = \ln(1 - e^{-x})$ 17. $\sqrt{2+\sqrt{x}}$

18. $j = \ln(\ln(x))$ 19. $P = \sqrt{1+\ln(x)}$ 20. $y = \sqrt[4]{s^3+1}$

(12) $\frac{d}{dx}(e^{-x^2}) = -2xe^{-x^2}$ (13) $3(x^2+4)^2 \cdot 2x = 6x(x^2+4)^2.$

(14) $\frac{5}{2t^2+3} \cdot 4t = \frac{20t}{2t^2+3}$ (15) $f'(\theta) = -(e^\theta + e^{-\theta})^{-2} \cdot (e^\theta - e^{-\theta})$

(16) $f'(x) = \frac{1}{1-e^{-x}} \cdot e^{-x}$ (17) $\frac{1}{2}(2+\sqrt{x})^{-1/2} \cdot (\frac{1}{2}x^{-1/2})$

(18) $\frac{dj}{dx} = \frac{1}{\ln x} \cdot \frac{1}{x} = \frac{1}{x\ln x}$ (19) $\frac{dP}{dx} = \frac{1}{2}(1+\ln x)^{-1/2} \cdot \frac{1}{x}$

(20) $\frac{dy}{ds} = \frac{1}{4}(s^3+1)^{-3/4} \cdot 3s^2$

21. If you invest P dollars in a bank account at an annual interest rate of $r\%$, then after t years you will have B dollars, where

$$B = P(1 + \frac{r}{100})^t.$$

(a) Find dB/dt, assuming P and r are constant. In terms of money, what does dB/dt represent?

(b) Find dB/dr, assuming P and t are constant. In terms of money, what does dB/dr represent?

(a) $\frac{dB}{dt} = P \cdot \ln(1+\frac{r}{100}) \cdot (1+\frac{r}{100})^t$ dollars/year. How quick your money is growing.

(b) $\frac{dB}{dr} = tP(1+\frac{r}{100})^{t-1} \cdot \frac{1}{100}$ dollars per percent. How much quicker your money would grow at a better interest rate.

22. Given $y = f(x)$ with $f(1) = 4$ and $f'(1) = 3$, find

(a) $g'(1)$ if $g(x) = \sqrt{f(x)}$.

(b) $h'(1)$ if $h(x) = f(\sqrt{x})$.

(a) $g'(x) = \frac{1}{2}(f(x))^{-1/2} \cdot f'(x)$

$g'(1) = \frac{1}{2}(4)^{-1/2} \cdot 3 = \frac{3}{4}$

(b) $h'(x) = f'(\sqrt{x}) \cdot \frac{1}{2}x^{-1/2}$

$h'(1) = 3 \cdot \frac{1}{2} = \frac{3}{2}$

Product and Quotient Rules: For functions $u = f(x)$ and $v = g(x)$ we have

$$(fg)' = f'g + fg' \qquad \text{and} \qquad (\frac{f}{g})' = \frac{f'g - fg'}{g^2}$$

or

$$\frac{d(uv)}{dx} = \frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx} \qquad \text{and} \qquad \frac{d(u/v)}{dx} = \frac{\frac{du}{dx} \cdot v - u \cdot \frac{dv}{dx}}{v^2}.$$

Exercises: Find the derivative of the given function unless otherwise specified. Assume a, b are constants.

23. $f(t) = te^{-2t}$ 24. $t^2 \ln t$ 25. $y = (t^2+3)e^t$
26. $w = x \cdot 2^x$ 27. $z = (3t+1)(5t+2)$ 28. $w = \frac{3y+y^2}{5+y}$
29. $y = \frac{1+z}{\ln z}$ 30. $f(x) = axe^{-bx}$ 31. $g(\alpha) = e^{\alpha e^{-2\alpha}}$

(23) $f'(t) = e^{-2t} + -2te^{-2t}$ (24) $\frac{d}{dt}(t^2 \ln t) = 2t\ln t + \frac{t^2}{t} = 2t \cdot \ln t + t$

(25) $\frac{dy}{dt} = 2te^t + (t^2+3)e^t$ (26) $\frac{dw}{dx} = 2^x + x \cdot \ln 2 \cdot 2^x$

(27) $\frac{dz}{dt} = 3(5t+2) + 5(3t+1)$ (28) $\frac{dw}{dy} = \frac{(5+y)(3+2y) - (3y+y^2)}{(5+y)^2}$

(29) $\frac{dy}{dz} = \frac{\ln z - \left(\frac{1+z}{z}\right)}{(\ln z)^2}$ (30) $f'(x) = ae^{-bx} + (-ab)xe^{-bx}$

(31) $g'(\alpha) =$ ~~[illegible]~~ $e^{\alpha e^{-2\alpha}}\left(e^{-2\alpha} + -2\alpha e^{-2\alpha}\right)$

32. If a person is lost in the wilderness, the search and rescue team identifies the boundaries of the search area and then uses probabilities to help optimize the chances of finding the person, assuming the subject is immobile. The probability, O, of the person being outside the search area after the search has begun and the person has not been found is given by

$$O = \frac{I}{1-(1-I)E},$$

where I is the probability of the person being outside the search area at the start of the search and E is the search effort, a measure of how well the search area has been covered by the resources in the field.

(a) If there was a 20% chance that the subject was not in the search area at the start of the search, and the search effort was 80%, what is the current probability of the person being outside the search area? (Probabilities are between 0 and 1, so 20%=0.2 and 80%=0.8.)

(b) In practical terms, what does $I = 1$ mean? Is this realistic?

(c) Evaluate $O'(E)$. Is it positive or negative? What does that tell you about O as E increases?

(c) $O'(E) = \frac{-I(-(1-I))}{(1-(1-I)E)^2} = \frac{I(1-I)}{(1-(1-I)E)^2} > 0$

(a) 55.56%

Thus as search effort increases, the probability the person is outside the search area increases.

(b) If $I=1$, then there is a 100% the person is outside the search area.